IU-MSTP/36
hep-lat/9906016
June, 1999

Axial Anomaly in Lattice Abelian Gauge Theory in Arbitrary Dimensions

Takanori Fujiwara, Hiroshi Suzuki and Ke Wu*

Department of Mathematical Sciences, Ibaraki University, Mito 310-8512, Japan

Abstract

Axial anomaly of lattice abelian gauge theory in hyper-cubic regular lattice in arbitrary even dimensions is investigated by applying the method of exterior differential calculus. The topological invariance, gauge invariance and locality of the axial anomaly determine the explicit form of the topological part. The anomaly is obtained up to a multiplicative constant for finite lattice spacing and can be interpreted as the Chern character of the abelian lattice gauge theory.

*On leave of absence from Institute of Theoretical Physics, Academia Sinica, P.O.Box 2735, Beijing 100080, China

Recent discovery of lattice Dirac operator [1, 2, 3] satisfying Ginsperg-Wilson (GW) relation [4] and implementing exact chiral symmetry [5] have opened up new possibility of understanding nonperturbative behaviors of chiral gauge theories on the lattice. The axial anomaly arises as the nontrivial Jacobian factor of the fermion measure [6] under the chiral transformations and is related to the index of the Dirac operator [2, 5]. Perturbative evaluation of the axial anomaly in the continuum limit was carried out in ref. [7] by using the overlap Dirac operator [3] and the axial anomaly of the contimuum theories were reproduced. See also refs. [8, 9]. Such explicit analysis becomes rather involved by its own right. However, it is plausable that the axial anomaly on the lattice is also related to some topological object as in continuum theory and its structure can be determined by invoking the method of differential geometry on the discrete lattice. In fact it was argued in ref. [10] that the topological invariance of the index of the GW Dirac operator and the gauge invariance almost fix the form of the axial anomaly, and the topologcal part of the anomaly is obtained up to a multiplicative constant for finite lattice spacing in four dimensions.

In this note we investigate the axial anomaly of lattice abelian gauge theory in euclidean hyper-cubic regular lattice in arbitrary even dimensions by applying the method of exterior differential calculus. We find that the topological invariance, gauge invariance and locality of the axial anomaly also determine the explicit form of the topological part in arbitrary dimensions. The axial anomaly is a natural extension of the result obtained in ref. [10] and has characteristic structure of products of field strengths contracted with the Levi-Civita symbol but each argument of the field strengths is shifted so that the axial anomaly acquires topological nature. We argue that such shifts in the arguments can be naturally understood within the framework of noncommutative differential calculus [11] and the axial anomaly is indeed the Chern character of abelian gauge theory on the discrete lattice.

Let us begin with some basic definitions in noncommutative differential calculus [11] on the hyper-cubic regular lattice $\mathbf{Z}^D$ of unit lattice spacing. See also ref. [10]. We denote the generators of exterior differential algebra by $\mathrm{d}x_\mu$ ($\mu = 1, \cdots, D$). They satisfy

$$\mathrm{d}x_\mu \mathrm{d}x_\nu = -\mathrm{d}x_\nu \mathrm{d}x_\mu \ , \qquad f(x)\mathrm{d}x_\mu = \mathrm{d}x_\mu f(x-\hat{\mu}) \ , \tag{1}$$

where $f(x)$ is an arbitrary function on $\mathbf{Z}^D$. In noncommutative differential calculus $\mathrm{d}x_\mu$ does not commute with the coordinates x_μ as in ordinary differential calculus. Instead, it generates a shift of the coordinates in the direction indicated by $\hat{\mu}$.

Differential k-forms on $\mathbf{Z}^D$ can be defined by

$$f = \frac{1}{k!} f_{\mu_1 \cdots \mu_k}(x) \mathrm{d}x_{\mu_1} \cdots \mathrm{d}x_{\mu_k} \ , \tag{2}$$

where $f_{\mu_1 \cdots \mu_k}(x)$ is completely antisymmetric in $\mu_1, \cdots, \mu_k$. The vector space of k-forms on $\mathbf{Z}^D$ is denoted by Ω_k.

On the discrete lattice one can introduce two kind of difference schemes, the forward and backward difference operators ∂_μ and ∂_μ^* defined by

$$\partial_\mu f(x) = f(x+\hat{\mu}) - f(x) \ , \qquad \partial_\mu^* f(x) = f(x) - f(x-\hat{\mu}). \tag{3}$$

Exterior differential operator $\mathrm{d}:\Omega_k \to \Omega_{k+1}$ on forms can be defined by the forward difference operator as

$$\mathrm{d}f = \frac{1}{k!}\partial_\mu f_{\mu_1\cdots\mu_k}(x)\mathrm{d}x_\mu \mathrm{d}x_{\mu_1}\cdots \mathrm{d}x_{\mu_k} \ . \tag{4}$$

Since two successive differences commute $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, the exetrior differential operator satisfies nilpotency relation $\mathrm{d}^2=0$. This enables us to define closed forms and exact forms as in ordinary exterior calculus.

The remarkable property of noncommutative differential calculus is that the Leibniz rule for the ordinary exterior differential calculus holds true because of the second property in (1). Let f and g be k- and l-forms, then one easily finds

$$\mathrm{d}(f(x)g(x)) = \mathrm{d}f(x)g(x) + (-1)^k f(x)\mathrm{d}g(x) \ . \tag{5}$$

We shall use another kind of exterior differential operator, the divergence operator $\mathrm{d}^*: \Omega_k \to \Omega_{k-1}$, defined by

$$\mathrm{d}^* f = \frac{1}{(k-1)!}\partial_\mu^* f_{\mu\mu_2\cdots\mu_k}(x)\mathrm{d}x_{\mu_2}\cdots \mathrm{d}x_{\mu_k} \ . \tag{6}$$

It also satisfies nilpotency $\mathrm{d}^{*2}=0$.

The nilpotency of the exterior difference operator naturally leads to an analog of Poincaré lemma. We quote it here from ref. [10]:

Lemma 1 *Let f be a closed k-form on $\mathbf{Z}^D$ with compact support and $\sum_x f(x) = 0$ for $k=D$, then there exists a $(k-1)$-form g such that $f = \mathrm{d}g$.*

Since d^* is also nilpotent, one can state Poincaré lemma in the following form:

Lemma 2 *Let f be a k-form on $\mathbf{Z}^D$ with compact support satisfying $\mathrm{d}^* f = 0$ and $\sum_x f(x) = 0$ for $k=0$, then there exists a $(k+1)$-form g such that $f = \mathrm{d}^* g$.*

Let us introduce another copy of the lattice $\mathbf{Z}^D$ and the generators of exterior algebra $\mathrm{d}y_\mu$ $(\mu = 1,\cdots,D)$ satisfying

$$\begin{aligned} &\mathrm{d}x_\mu \mathrm{d}y_\nu = -\mathrm{d}y_\nu \mathrm{d}x_\mu \ , \qquad \mathrm{d}y_\mu \mathrm{d}y_\nu = -\mathrm{d}y_\nu \mathrm{d}y_\mu \ , \\ &f(x,y)\mathrm{d}x_\mu = \mathrm{d}x_\mu f(x-\hat{\mu},y) \ , \qquad f(x,y)\mathrm{d}y_\mu = \mathrm{d}y_\mu f(x,y-\hat{\mu}) \ , \end{aligned} \tag{7}$$

where f is an arbitrary function of $(x, y) \in \mathbf{Z}^D \times \mathbf{Z}^D$. We define difference operators by

$$\begin{aligned} \partial_\mu f(x,y) &= f(x+\hat{\mu},y) - f(x,y) \ , \qquad & f(x,y)\overleftarrow{\partial}_\mu &= f(x,y+\hat{\mu}) - f(x,y) \ , \\ \partial^*_\mu f(x,y) &= f(x,y) - f(x-\hat{\mu},y) \ , \qquad & f(x,y)\overleftarrow{\partial}^*_\mu &= f(x,y) - f(x,y-\hat{\mu}) \ . \end{aligned} \tag{8}$$

A differential (k, l)-form on $\mathbf{Z}^D \times \mathbf{Z}^D$ is defined by

$$f = \frac{1}{k!l!} f_{\mu_1\cdots\mu_k|\nu_1\cdots\nu_l}(x,y) \mathrm{d}x_{\mu_1}\cdots \mathrm{d}x_{\mu_k}\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_l} \ , \tag{9}$$

where $f_{\mu_1\cdots\mu_k|\nu_1\cdots\nu_l}(x,y)$ is completely antisymmetric in $\mu_1, \cdots, \mu_k$ and in $\nu_1, \cdots, \nu_l$, separately, and is assumed to have compact support on $\mathbf{Z}^D \times \{y\}$ and $\{x\} \times \mathbf{Z}^D$. We denote the vector space of (k, l)-forms by $\Omega_{k,l}$.

The exterior differential with respect to x or y is denoted by d_x or d_y. For the (k, l)-form (9) they satisfy

$$\begin{aligned} \mathrm{d}_x f &= \frac{1}{k!l!}\partial_\mu f_{\mu_1\cdots\mu_k|\nu_1\cdots\nu_l}(x,y)\mathrm{d}x_\mu \mathrm{d}x_{\mu_1}\cdots\mathrm{d}x_{\mu_k}\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_l} \ , \\ \mathrm{d}_y f &= \frac{(-1)^k}{k!l!} f_{\mu_1\cdots\mu_k|\nu_1\cdots\nu_l}(x,y)\overleftarrow{\partial}_\nu \mathrm{d}x_{\mu_1}\cdots\mathrm{d}x_{\mu_k}\mathrm{d}y_\nu \mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_l} \ . \end{aligned} \tag{10}$$

Divergence operators (6) can also be extended to (k, l)-forms:

$$\begin{aligned} \mathrm{d}^*_x f &= \frac{1}{(k-1)!l!}\partial^*_\mu f_{\mu\mu_2\cdots\mu_k|\nu_1\cdots\nu_l}(x,y)\mathrm{d}x_{\mu_2}\cdots\mathrm{d}x_{\mu_k}\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_l} \ , \\ \mathrm{d}^*_y f &= \frac{(-1)^k}{k!(l-1)!} f_{\mu_1\cdots\mu_k|\nu\nu_2\cdots\nu_l}(x,y)\overleftarrow{\partial}^*_\nu \mathrm{d}x_{\mu_1}\cdots\mathrm{d}x_{\mu_k}\mathrm{d}y_{\nu_2}\cdots\mathrm{d}y_{\nu_l} \ . \end{aligned} \tag{11}$$

It is straightforward to show that these operators satisfy nilpotency relations

$$\mathrm{d}_x^2 = \mathrm{d}_y^2 = (\mathrm{d}_x + \mathrm{d}_y)^2 = 0 \ , \qquad \mathrm{d}_x^{*2} = \mathrm{d}_y^{*2} = (\mathrm{d}^*_x + \mathrm{d}^*_y)^2 = 0 \ . \tag{12}$$

A differential form ω satisfying

$$\mathrm{d}^*_x \mathrm{d}^*_y \omega(x,y) = 0 \tag{13}$$

is of special interest because it gives rise to a sequence of forms related by descent equations. Later we encounter $(2m, 2)$-forms of this type in analyzing axial anomaly in abelian lattice gauge theory. To be definite let us consider a (k, l)-form $\omega^{k,l}$ satisfying (13), then by the Poincaré lemma there exist forms $\omega^{k\pm1,l\mp1}$ satisfying

$$\mathrm{d}^*_x \omega^{k+1,l-1}(x,y) + \mathrm{d}^*_y \omega^{k,l}(x,y) = 0 \ , \qquad \mathrm{d}^*_x \omega^{k,l}(x,y) + \mathrm{d}^*_y \omega^{k-1,l+1}(x,y) = 0 \ . \tag{14}$$

Since these forms also satisfy (13), they lead to new forms $\omega^{k\pm2,l\mp2}$. Such procedure can be continued until one ends up with $\omega^{k+l,0}$ and $\omega^{0,k+l}$ for $k+l \leq D$ or $\omega^{D,k+l-D}$ and $\omega^{k+l-D,D}$ for $k+l > D$. If we define a formal sum of differential forms

$$\omega = \sum_{j=0}^{\min\{k+l,D\}} \omega^{k+l-j,j} \ , \tag{15}$$

then the descent equations can be compactly expressed as

$$(\mathrm{d}^*_x + \mathrm{d}^*_y)\omega(x,y) = 0 \ . \tag{16}$$

We restrict ourselves to $m \equiv k+l \leq D$ and solve the descent equations. As we shall see, only the forms with $m \leq D$ appear in analyzing the axial anomaly.

We first define an m-form α^m on $\mathbf{Z}^D$ by

$$\begin{aligned} \alpha^m(y) &\equiv \sum_x \omega^{0,m}(x,y) \\ &\equiv \frac{1}{m!}\alpha_{\nu_1\cdots\nu_m}(y)\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_m} \end{aligned} \tag{17}$$

This can be shown to be divergence free

$$\mathrm{d}^*_y\alpha^m(y) = 0 \tag{18}$$

by the descent equation $\mathrm{d}^*_y\omega^{0,m}(x,y) = -\mathrm{d}^*_x\omega^{1,m-1}(x,y)$. The key observation in solving the descent equations is that $\omega^{0,m}(x,y)$ can be decomposed as

$$\omega^{0,m}(x,y) = \alpha^{0,m}(x,y) + \mathrm{d}^*_x\vartheta^{1,m}(x,y) \ , \tag{19}$$

where we have introduced $\alpha^{0,m}(x,y) \equiv \delta_{x,y}\alpha^m(y)$ with $\delta_{x,y}$ the Kronecker δ-symbol and $\vartheta^{1,m}(x,y)$ is some form in $\Omega_{1,m}$. This can be shown by applying the Poincaré lemma for $\omega^{0,m}(x,y) - \alpha^{0,m}(x,y)$ as a 0-form on $\mathbf{Z}^D \times \{y\}$. The descent equation $\mathrm{d}^*_x\omega^{1,m-1} + \mathrm{d}^*_y\omega^{0,m} = 0$ can then be solved as follows

$$\begin{aligned} \mathrm{d}^*_x\omega^{1,m-1} &= -\mathrm{d}^*_y\omega^{0,m} \\ &= -\mathrm{d}^*_y\alpha^{0,m}(x,y) + \mathrm{d}^*_x\mathrm{d}^*_y\vartheta^{1,m}(x,y) \\ &= \mathrm{d}^*_x\alpha^{1,m-1}(x,y) + \mathrm{d}^*_x\mathrm{d}^*_y\vartheta^{1,m}(x,y) \ , \end{aligned} \tag{20}$$

where we have defined a $(1,m-1)$-form $\alpha^{1,m-1}(x,y)$ by

$$\alpha^{1,m-1}(x,y) \equiv \frac{1}{(m-1)!}\alpha_{\mu\nu_1\cdots\nu_{m-1}}(x)\delta_{x,y-\hat{\mu}}\mathrm{d}x_\mu\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_{m-1}} \ . \tag{21}$$

The key relation $\mathrm{d}_y^*\alpha^{0,m}(x,y) = -\mathrm{d}_x^*\alpha^{1,m-1}(x,y)$ can be seen as follows. First note that it can be explicitly written as

$$\begin{aligned}&\frac{1}{(m-1)!}(\delta_{x,y}\alpha_{\mu\nu_1\cdots\nu_{m-1}}(y) - \delta_{x,y-\hat{\mu}}\alpha_{\mu\nu_1\cdots\nu_{m-1}}(y-\hat{\mu}))\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_{m-1}}\\ &= -\frac{1}{(m-1)!}(\alpha_{\mu\nu_1\cdots\nu_{m-1}}(x)\delta_{x,y-\hat{\mu}} - \alpha_{\mu\nu_1\cdots\nu_{m-1}}(x-\hat{\mu})\delta_{x-\hat{\mu},y-\hat{\mu}})\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_{m-1}}\ . \quad (22)\end{aligned}$$

This holds true if one notices the divergence free condition (18), i.e.,

$$\partial_\mu^*\alpha_{\mu\nu_1\cdots\nu_{m-1}}(x) = \alpha_{\mu\nu_1\cdots\nu_{m-1}}(x) - \alpha_{\mu\nu_1\cdots\nu_{m-1}}(x-\hat{\mu}) = 0\ . \quad (23)$$

We see from (20) that $\omega^{1,m-1}$ can be expressed as

$$\omega^{1,m-1}(x,y) = \alpha^{1,m-1}(x,y) + \mathrm{d}_x^*\vartheta^{2,m-1}(x,y) + \mathrm{d}_y^*\vartheta^{1,m}(x,y) \quad (24)$$

for some form $\vartheta^{2,m-1}$ in $\Omega_{2,m-1}$. This procedure of solving the descent equations can be carried out until all the froms $\omega^{j,m-j}$ $(j = 0,\cdots,m)$ are obtained. One can easily convince himself that $\omega^{k,l}$ is given by

$$\omega^{k,l}(x,y) = \alpha^{k,l}(x,y) + \mathrm{d}_x^*\vartheta^{k+1,l}(x,y) + \mathrm{d}_y^*\vartheta^{k,l+1}(x,y)\ , \quad (25)$$

where $\vartheta^{k+1,l}(x,y) \in \Omega_{k+1,l}$ and $\vartheta^{k,l+1}(x,y) \in \Omega_{k,l+1}$ are some forms on $\mathbf{Z}^D \times \mathbf{Z}^D$ and $\alpha^{k,l}(x,y)$ is defined by

$$\alpha^{k,l}(x,y) \ \equiv\ \frac{1}{k!l!}\alpha_{\mu_1\cdots\mu_k\nu_1\cdots\nu_l}(x)\delta_{x,y-\hat{\mu}_1-\cdots-\hat{\mu}_k}\mathrm{d}x_{\mu_1}\cdots\mathrm{d}x_{\mu_k}\mathrm{d}y_{\nu_1}\cdots\mathrm{d}y_{\nu_l}\ . \quad (26)$$

It is straightforward to show that $\alpha^{k,l}$ satisfy the descent equations (14).

We now turn to the analysis of axial anomaly of abelian lattice gauge theory on $\mathbf{Z}^D$. We shall use gauge potential $A(x) = A_\mu(x)\mathrm{d}x_\mu$ instead of link variable $U_\mu(x) = \exp iA_\mu(x)$. The field strength $F(x) = \mathrm{d}A(x) = \frac{1}{2}F_{\mu\nu}(x)\mathrm{d}x_\mu\mathrm{d}x_\nu$ can be identified with the usual definition $-i\ln[U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^{-1}U_\nu(x)^{-1}]$ if field configurations satisfy

$$\sup_{x,\mu,\nu}|F_{\mu\nu}(x)| < \epsilon \qquad \left(0 < \epsilon < \frac{\pi}{3}\right)\ . \quad (27)$$

We assume that the gauge field configurations are subject to this admissibility condition [10].

The axial anomaly $q(x)$ is expressed by the GW Dirac operator $D(x,y) \equiv D(x)\delta_{x,y}$ as [2, 5]

$$q(x) \ =\ \mathrm{tr}\,\gamma_5\left[1 - \frac{1}{2}D(x)\right]\delta_{x,x}\ . \quad (28)$$

It is assumed to depend locally on the gauge potential and is gauge invariant. Furthermore, the sum of $q(x)$ over the lattice is the index of the GW Dirac operator and is topologically invariant. This implies under an arbitrary local variation of gauge potential $\delta A_\mu(x)$

$$\sum_x \delta q(x) \quad = \quad 0 \ . \tag{29}$$

These properties largely restrict the structure of the axial anomaly as argued in ref. [10]. In fact the generic structure of the axial anomaly is obtained in four dimensions. This result can be extend to arbitrary even dimensions $D = 2n$. We state this in the following theorem:

Theorem *Let $q(x)$ be a gauge invariant function which depends locally on the gauge potential and the sum over the lattice is a topological invariant, then it can be expressed as*

$$\begin{aligned} q(x) \quad &= \quad \left(\frac{1}{2}\right)^n C\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n} F_{\mu_1\nu_1}(x) F_{\mu_2\nu_2}(x+\hat\mu_1+\hat\nu_1) \\ &\quad \times\cdots\times F_{\mu_n\nu_n}(x+\hat\mu_1+\hat\nu_1+\cdots+\hat\mu_{n-1}+\hat\nu_{n-1}) + \partial^*_\mu k_\mu(x) \ , \end{aligned} \tag{30}$$

where C is a constant, $\epsilon_{\mu_1\cdots\mu_D}$ is the Levi-Civita symbol in D dimensions and $k_\mu(x)$ is a gauge invariant current.

Before turning to the proof of this theorem, we give here a few remarks. We first note the connection of the topological term $q(x) - \partial^*_\mu k_\mu(x)$ with the noncommutative differential calculus. The shifts of the arguments of the field strengths in (30) are very important. It can naturally be interpreted as the Chern character of abelian gauge theory on the lattice

$$\begin{aligned} C(F(x))^n \quad &= \quad \left(\frac{1}{2}\right)^n C\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n} F_{\mu_1\nu_1}(x) F_{\mu_2\nu_2}(x+\hat\mu_1+\hat\nu_1) \\ &\quad \times\cdots\times F_{\mu_n\nu_n}(x+\hat\mu_1+\hat\nu_1+\cdots+\hat\mu_{n-1}+\hat\nu_{n-1})\mathrm{d}^D x \ , \end{aligned} \tag{31}$$

where $\mathrm{d}^D x \equiv \mathrm{d}x_1\cdots\mathrm{d}x_D$ is the volume form and the arguments of $F_{\mu\nu}$ are shiftted by the noncommutativity nature (1). Secondly, the anomaly coefficient C can be found easily from the well-known results in continuum theories [9]. In the case of abelian chiral gauge theory the gauge anomaly also takes the form (30) and should be cancelled to guarantee the gauge invariance. In anomaly free theories the topological term is absent but the second term in (30) may be nonvanishing. It represents an artificial breaking of the gauge symmetry due to the finite lattice spacing and can be removed by a suitable choice of the path integral measure [12].

To verify the theorem let us consider a gauge invariant $2m$-form α^{2m} on $\mathbf{Z}^D$ which depends locally on the gauge potential and satisfy $\mathrm{d}^*\alpha^{2m} = 0$, then its coefficients can be expressed as

$$\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x) = \beta_{\mu_1\nu_1\cdots\mu_m\nu_m} + \sum_y \omega_{\mu_1\nu_1\cdots\mu_m\nu_m|\rho}(x,y)A_\rho(y) \ , \tag{32}$$

where $\omega_{\mu_1\nu_1\cdots\mu_m\nu_m|\rho}(x,y)$ are the coefficients of a gauge invariant $(2m,1)$-form $\omega^{2m,1}(x,y)$ defined by (9) and are explicitly given by

$$\omega_{\mu_1\nu_1\cdots\mu_m\nu_m|\rho}(x,y) \quad = \quad \int_0^1 dt \left(\frac{\partial \alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x)}{\partial A_\rho(y)} \right)_{A\to tA} . \tag{33}$$

The $\beta_{\mu_1\nu_1\cdots\mu_m\nu_m}$ is the field independent part of $\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x)$. The dependences of $\beta_{\mu_1\nu_1\cdots\mu_m\nu_m}$ on the lattice coordinates can be excluded by assuming the translation invariance of $q(x)$, i.e., it depends on the lattice coordinates through the gauge potential or the coordinate differences.

The gauge invariance of α^{2m} and the divergence free condition $\mathrm{d}^*\alpha^{2m} = 0$ imply

$$\mathrm{d}^*_x\omega^{2m,1} = \mathrm{d}^*_y\omega^{2m,1} = 0 \ . \tag{34}$$

By the Poincaré lemma we see that there exist a form $\omega^{2m,2}(x,y)$ such that

$$\omega^{2m,1}(x,y) \quad = \quad -\mathrm{d}^*_y\omega^{2m,2}(x,y) \ , \qquad \mathrm{d}^*_x\mathrm{d}^*_y\omega^{2m,2}(x,y) \quad = \quad 0 \ . \tag{35}$$

The first of these relations implies that (32) can be expressed by the field strength as

$$\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x) = \beta_{\mu_1\nu_1\cdots\mu_m\nu_m} + \frac{1}{2}\sum_y \omega_{\mu_1\nu_1\cdots\mu_m\nu_m|\rho\sigma}(x,y)F_{\rho\sigma}(y) \ , \tag{36}$$

while the second one leads to the decomposition of $\omega^{2m,2}(x,y)$ as in (25)

$$\omega^{2m,2}(x,y) = \alpha^{2m,2}(x,y) + \mathrm{d}^*_x\vartheta^{2m+1,2}(x,y) + \mathrm{d}^*_y\vartheta^{2m,3}(x,y) \ . \tag{37}$$

The forms $\alpha^{2m,2}$, $\vartheta^{2m+1,2}$ and $\vartheta^{2m,3}$ are all gauge invariant [10] and in particular $\alpha^{2m,2}$ can be found from (26) for $k = 2m$ and $l = 2$ with some gauge invariant $(2m+2)$-form α^{2m+2}. Note that α^{2m+2} is also divergence free as is α^{2m}. It is now straightfoward to show that $\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x)$ is given by

$$\begin{aligned} \alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x) \quad = \quad & \beta_{\mu_1\nu_1\cdots\mu_m\nu_m} + \frac{1}{2}\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m\rho\sigma}(x)F_{\rho\sigma}(x+\hat\mu_1+\hat\nu_1+\cdots+\hat\mu_m+\hat\nu_m) \\ & +\partial^*_\mu\left(\frac{1}{2}\sum_y \vartheta_{\mu\mu_1\nu_1\cdots\mu_m\nu_m|\rho\sigma}(x,y)F_{\rho\sigma}(y)\right) . \end{aligned} \tag{38}$$

The field independent constant term can be cast into a total divergence for $m < n$ by introducing

$$\bar\beta_{\mu\mu_1\nu_1\cdots\mu_m\nu_m}(x) \quad = \quad x_{[\mu}\beta_{\mu_1\nu_1\cdots\mu_m\nu_m]} \ , \tag{39}$$

where $[\cdots]$ stands for the antisymmetrization of indices. The case $m = n$ is special. In this case $\beta_{\mu_1\nu_1\cdots\mu_n\nu_n}$ is proportional to $\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}$. Furthermore, we can not convert it to a total divergence of some $(D+1)$-form. We thus obtain

$$\begin{aligned}
\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m}(x) &= \frac{1}{2}\alpha_{\mu_1\nu_1\cdots\mu_m\nu_m\rho\sigma}(x)F_{\rho\sigma}(x+\hat{\mu}_1+\hat{\nu}_1+\cdots+\hat{\mu}_m+\hat{\nu}_m) \\
&\quad +\partial^*_\mu\bar{\vartheta}_{\mu\mu_1\nu_1\cdots\mu_m\nu_m}(x) \qquad \text{for} \qquad m<n\ , \\
\alpha_{\mu_1\nu_1\cdots\mu_n\nu_n}(x) &= C\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n} \qquad \text{for} \qquad m=n\ ,
\end{aligned} \tag{40}$$

where $\bar{\vartheta}_{\mu\mu_1\nu_1\cdots\mu_m\nu_m}(x)$ is given by

$$\bar{\vartheta}_{\mu\mu_1\nu_1\cdots\mu_m\nu_m}(x) \equiv \frac{2m+1}{D-2m}\bar{\beta}_{\mu\mu_1\nu_1\cdots\mu_m\nu_m}(x)+\frac{1}{2}\sum_y \vartheta_{\mu\mu_1\nu_1\cdots\mu_m\nu_m|\rho\sigma}(x,y)F_{\rho\sigma}(y)\ . \tag{41}$$

It is completely antisymmetric in its indices and is gauge invariant. Eq. (40) serves as a recurrence relation in expanding $q(x)$ in terms of the gauge potential.

We now turn to the proof of the theorem. The general argument given above allow us to expand $q(x)$ as

$$q(x) = \frac{1}{2}\alpha_{\mu\nu}(x)F_{\mu\nu}(x)+\partial^*_\lambda\bar{\vartheta}_\lambda(x)\ , \tag{42}$$

where a 2-form $\alpha^2 \equiv \frac{1}{2}\alpha_{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu$ satisfies $\mathrm{d}^*\alpha^2 = 0$ as can be easily seen from the topological invariance (29). So we can apply (40) also for $\alpha_{\mu\nu}(x)$. This leads to

$$\begin{aligned}
q(x) &= \left(\frac{1}{2}\right)^2\alpha_{\mu_1\nu_1\mu_2\nu_2}(x)F_{\mu_1\nu_1}(x)F_{\mu_2\nu_2}(x+\hat{\mu}_1+\hat{\nu}_1) \\
&\quad +\partial^*_\lambda\left(\bar{\vartheta}_\lambda(x)+\frac{1}{2}\bar{\vartheta}_{\lambda\mu\nu}(x)F_{\mu\nu}(x+\hat{\lambda})\right)\ .
\end{aligned} \tag{43}$$

In deriving this we have used the relation

$$\partial^*_\lambda(\bar{\vartheta}_{\lambda\mu\nu}(x)F_{\mu\nu}(x+\hat{\lambda})) = \partial^*_\lambda\bar{\vartheta}_{\lambda\mu\nu}(x)F_{\mu\nu}(x)+\bar{\vartheta}_{\lambda\mu\nu}(x)\partial_\lambda F_{\mu\nu}(x) \tag{44}$$

and the Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$. Again a 4-form $\alpha^4 \equiv \frac{1}{4!}\alpha_{\mu\nu\rho\sigma}\mathrm{d}x_\mu\mathrm{d}x_\nu\mathrm{d}x_\rho\mathrm{d}x_\sigma$ satisfies $\mathrm{d}^*\alpha^4 = 0$, and the current appearing in the second line of (43) is gauge invariant. Obviousely, this procedure can be repeated until we end up with (30). This completes the proof of the theorem.

In conclusion we have obtained the axial anomaly of lattice abelian gauge theory by extending the arguments of ref. [10] to arbitrary even dimensions. It is given by the Chern character of abelian gauge theory on the lattice. This result can also be achieved by the systematic BRST analysis [13] extended to lattice gauge theory. In our present treatment the role of noncommutativity nature (1) is somewhat indirect. But in the BRST analysis noncommutative differential calculus plays an essential role.

References

[1] P. Hasenfratz, Nucl. Phys. (Proc. Suppl.) **63** (1998) 53; Nucl. Phys. **B525** (1998) 401.

[2] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. **B427** (1998) 125.

[3] H. Neuberger, Phys. Lett. **B417** (1998) 141; **B427** (1998) 353.

[4] P.H. Ginsparg and K.G. Wilson, Phys. Rev. **D25** (1982) 2649.

[5] M. Lüscher, Phys. Lett. **B428** (1998) 342.

[6] K. Fujikawa, Phys. Rev. Lett. **42** (1979) 1195; Phys. Rev. **D21** (1980) 2848; **D22** (1980) 1499(E); **D29** (1984) 285.

[7] Y. Kikukawa and A. Yamada, Phys. Lett. **B448** (1999) 265.

[8] D.H. Adams, hep-lat/9812003;
H. Suzuki, hep-th/9812019, to appear in Prog. Theor. Phys.

[9] K. Fujikawa, Nucl. Phys. **B546** (1999) 480.

[10] M. Lüscher, Nucl. Phys. **B538** (1999) 515.

[11] A. Connes, "Noncommutative Geometry", Academic Press, New York, 1994;
A. Sitarz, J. Geom. Phys. **15** (1995) 123;
H.G. Ding, H.Y. Guo, J.M. Li and K. Wu, Z. Phys. **C64** (1994) 521;
A. Dimikis, F. Muller-Hoissen and T. Striker, J. Phys. A: Math. Gen. **26** (1994) 1927.

[12] M. Lüscher, Nucl. Phys. **B549** (1999) 295.

[13] T. Fujiwara, H. Suzuki and K. Wu, Ibaraki University preprint IU-MSTP/35, hep-lat/9906015.